International Journal of Theoretical Physics, Vol. 15, No. 5 (1976), pp. 319–321

A Simple Application of the Newman–Penrose Spin Coefficient Formalism. II

TALMADGE M. DAVIS

Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29631[1]

Received: 9 *June* 1975

Abstract

This is the second paper giving a simple application of the Newman–Penrose formalism to be used by those learning this theory. Here we solve the spherically symmetric Einstein–Maxwell problem for the Reissner-Nordstrøm solution using the Newman–Penrose formalism. As in our previous example the calculation is carried out using a Minkowski null tetrad.

As another simple application of the Newman–Penrose (NP) spin coefficient formalism (Newman and Penrose, 1962) [see our previous paper (Davis, 1976) for the Schwarzschild solution] we present the Einstein-Maxwell solution for the case of static spherical symmetry. This will yield the Reissner-Nordstrøm solution, which is equivalent to the general, time-dependent solution via a Birkhoff-type theorem.

With the hope that the reader has a grasp of the basics of the NP formalism [see Davis (1976) for a brief discussion], we will not present them here. The notation is that of Newman and Penrose (1962) and Davis (1976).

The metric for static spherical symmetry can be written

$$ds^2 = e^{2v}dt^2 - e^{2u}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where v and u depend only on r. Since we will perform our calculations in the orthonormal frame

$$\begin{aligned} \omega^1 &= e^u dr \\ \omega^2 &= r\,d\theta \\ \omega^3 &= r\sin\theta\, d\phi \\ \omega^4 &= e^v\, dt \end{aligned} \tag{2}$$

the metric takes the form of the Minkowski flat-space metric.

[1] Dr. Davis' present address is Department of Math and Physics, Cumberland College, Williamsburg, Kentucky 40769.

Calculations in the NP formalism are based upon a tetrad of null vectors which we will choose to be

$$
\begin{aligned}
l_\mu &= (\delta_\mu{}^1 + \delta_\mu{}^4)/\sqrt{2} \\
n_\mu &= (-\delta_\mu{}^1 + \delta_\mu{}^4)/\sqrt{2} \\
m_\mu &= (\delta_\mu{}^2 + i\delta_\mu{}^3)/\sqrt{2} \\
\overline{m}_\mu &= (\delta_\mu{}^2 - i\delta_\mu{}^3)/\sqrt{2}
\end{aligned}
\tag{3}
$$

The use of this Minkowski null tetrad is important in this spin coefficient method of finding solutions for a given metric. Since the metric is Minkowskiam when one does the calculation in an orthonormal frame, the null tetrad (3) can always be used as a starting point for the calculations. Using this tetrad we find the spin coefficients to be

$$\kappa = \sigma = \tau = \nu = \lambda = \pi = 0 \tag{4}$$

$$\rho = \mu = e^{-u}/\sqrt{2}r \tag{5}$$

$$\alpha = -\beta = \text{cotan}\ \theta/2\sqrt{2}r \tag{6}$$

$$\epsilon = \gamma = -(1/2\sqrt{2})v,_1\ e^{-u} \tag{7}$$

where the comma denotes ordinary differentiation with respect to r.

From the symmetry of the space-time we see that only radial electric and magnetic fields are present. Maxwell's equations eliminate the radial magnetic field; therefore, the tetrad components of the field tensor are

$$\phi_0 \equiv F_{\mu\nu} l^\mu m^\nu = 0 \tag{8}$$

$$\phi_2 \equiv F_{\mu\nu} \overline{m}^\mu n^\nu = 0 \tag{9}$$

$$\text{Im}\ \phi_1 \equiv \tfrac{1}{2} F_{\mu\nu} \overline{m}^\mu m^\nu = 0 \tag{10}$$

$$\text{Re}\ \phi_1 \equiv \tfrac{1}{2} F_{\mu\nu} l^\mu n^\nu \neq 0 \tag{11}$$

The Einstein equations in the NP formalism are written in terms of the tetrad components of the Ricci tensor, Φ_{ab}. From the Einstein-Maxwell equations we have

$$\Phi_{ab} = 8\pi T_{ab} \tag{12}$$

where

$$T_{ab} = \phi_a \overline{\phi}_b / 4\pi \tag{13}$$

(the bar denotes the complex conjugate) we find the Ricci tensor is

$$\Phi_{ab} = 2\phi_a \overline{\phi}_b \tag{14}$$

We have used geometrized units $G = c = 1$. Substituting Re $\phi_1 \equiv \phi$ into the energy-momentum tensor, we have

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{12} = \Phi_{22} = 0 \tag{15}$$

$$\Phi_{11} = 2\phi^2 \tag{16}$$

As shown in Davis (1976), all spherically symmetric space-times are Petrov type D, so the tetrad components of the Weyl tensor are

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{17}$$

$$\Psi_2 \neq 0 \tag{18}$$

Now, the nontrivial NP equations are

$$\rho_{,1} = -\sqrt{2}e^u(\rho^2 + 2\epsilon\rho) \tag{19}$$

$$\epsilon_{,1} = -e^u(\Psi_2 + 2\phi^2 - 4\epsilon^2)/\sqrt{2} \tag{20}$$

$$\phi^2 = (\Psi_2 - \rho^2 + \tfrac{1}{2}r^2)/2 \tag{21}$$

$$\Psi_2 = 4\epsilon\rho \tag{22}$$

The nontrivial Maxwell equation is

$$\phi_{,1} = -2\sqrt{2}\, e^u \rho\phi \tag{23}$$

Substituting (5) into (23) and solving the differential equation, we have

$$\phi = k/r^2 \tag{24}$$

which we see from (11) is equal to one-half the electric field. The constant k then is one-half the total charge Q, so

$$\phi = Q/2r^2 \tag{25}$$

Integration of the differential equation (19) yields

$$(u + v)_{,1} = 0 \tag{26}$$

Substitution of (22) and (26) into (21) gives

$$e^{-2u} = e^{2v} = 1 - r_0/r + Q^2/r^2 \tag{27}$$

The constant r_0, evaluated at large distance and for $Q = 0$, is twice the mass M of the gravitating body. Therefore, the metric is

$$\begin{aligned} ds^2 = (1 - 2M/r + Q^2/r^2)\, dt^2 - (1 - 2M/r + Q^2/r^2)^{-1}\, dr^2 \\ - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{28}$$

which is the Reissner-Nordstrøm metric.

References

Davis, T. M. (1976). This issue, *International Journal of Theoretical Physics*, **15**, 315.
Newman, E. T. and Penrose, R. (1962). *Journal of Mathematical Physics*, **3**, 566.